LIMIT LOAD ANALYSIS AND DESIGN OF AXISYMMETRIC PLATES UNDER THERMAL AND MECHANICAL LOADS

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C E R T I F I C A T E

This is to certify that the thesis entitled 'Limit load analysis of axisymmetric plates under thermal and mechanical loads' by K. UMESH is a record of work carried out under my supervision and that it has not been submitted elsewhere for a degree.

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CONTENTS

LIMIT LOAD ANALYSIS AND DESIGN OF AXISYMMETRIC PLATES UNDER THERMAL AND MECHANICAL LOADS

ABSTRACT

Many componentsof the neuclear reactors, thermal reactors, aerospace structures can be idealised as plate and shell structures. Under working conditions they are usually subjected to elevated temperatures and also called upon to resist mechanical loads. The Moment capacities or the ultimate load capacities of these structures are dependent on the material and geometric properties. It becomes worthwhile to study the change in the behaviour of such structures under those environments.

As a first step the present investigation is confined to axisymmetric plates with different boundary conditions. To this end an extensive literature survey is made and general equations are derived taking different temperature drop laws across the thickness of the plate, and in the plane of the plate, and taking hypothetical temperature range. Differential equations are solved using the toolsof Numerical Analysis. Materials are assumed to obey Tresca's yield criterion. Illustrative design problems are solved.

Design examples have been illustrated based on the results of this investigation. They can be used as ready reckoners for the design of such slabs. This investigation can be extended to different yield criteria and also to shallow shells.

NOMENCLATURE

The following nomenclature has been used in the thesis unless otherwise mentioned.

C	Half thickness of the plate.
C_{11}, C_{21}, C_{12}, C_{22} -	Constants of integration.
E -	Youngs modulas.
K_t -	Constant thermal loading.
M_t -	Thermal moment.
M_r -	Radial moment.
M_θ -	Circumferential moment.
M_0 -	Moment capacity.
$p(r)$ -	Mechanical load distribution.
R -	Radius of the plate.
r -	Radial co-ordinate.
T_1, T_2	Temperatures.
T_{b1}, T_{b2}	Temperatures on the bottom surface at the centre at the circumference respectively.
T_{t1}, T_{t2}	Temperatures on the top surface at the centre & at the circumference respectively.

Z	-	Co-ordinate axis.
α	-	Co-efficient of thermal expansion.
∇^2	-	$\left(\frac{d^2}{dr^2} + \frac{1}{r} \cdot \frac{d}{dr}\right)$
ν	-	Poissons ratio.

CHAPTER 1

INTRODUCTION

A structure can be defined as an assemblage of elements called upon to resist safely the external forces coming upon it and transmit them to the supports. A linear or discrete structural element is one in which one of the three dimensions is predominant than the other two. A continuum structure is one in which two of the dimensions are comparatively greater than the third, which is usually the thickness. If an example of the former case can be cited as a beam or a truss element, it can be a plate or a shell for the latter case. External forces can be in many forms. It may be in the form of contact pressure, it may be the pressure of a fluid flowing past the structure or it may be the thermal stresses induced due to temperature etc.,

In the case of, thermal nuclear reactors, aero-space structures and the like design of structural components for thermal effects merits equal consideration as for the external loads. The thermal effects are so severe that they consume a considerable part

of the load carring capacity of those structures. It becomes therefore worthwhile to study the effect of thermal loads on their limit state capacities. Many of these structures can be idealised as plates or shallow shell structures with suitable boundary conditions. Moment equations for the limit state condition for these elements are available in the literature. Of these, the lower bound techniques merit consideration from the view point of safety.

The modification of plate equations to take into account the effect of temperature, have not very much attracted the attention of researchers. This investigation aims, as a first step, to analyse the strength of plate elements subjected to a rather severe thermal environment. In arriving at the limit load capacities, the YIELD EQUALITY methoddeveloped by ADIDAM (Ref.10) will be included as load terms in the general differential equation. It is worthwhile to note that the initial stresses and strains have no effect on the moment capacity of the structure. The temperature drop across the thickness is a matter subjected to considerable difference of opinion. Various drop laws will be considered in this investigation and their effects studied. Normally as heat moves from source outwards it gives rise to different temperaturesat different points in the plane of the

plate. As a further step this effect will also be considered in the general equations derived. The equations will be specialised for various boundary conditions. Some real life material is also considered and design charts will be prepared for different temperature ranges, drop laws and boundary conditions.

A critical review of the existing literature on thermal stresses of plates is presented in Chapter II. In Chapter III general expressions for the moments due to temperature drops across the thickness of the plate considering various drop laws are derived. The most general form of plate equation incorporating temperature terms, for the limit state of the plate is derived. The yield equality method and the general solution of the plate equation for limit state is solved incorporating the temperature terms. They are also specialised for different boundary conditions and are given in Chapter IV. In Chapter V application of the limit design developed is illustrated with the aid of typical example problems.

Suitable conclusions are drawn based on the results of the investigation and the scope for further work is also indicated in Chapter VI.

CHAPTER II

LITERATURE SURVEY

MAULBETSCH (Ref. 1) presented solutions of the problem of thermal stresses in plates with simply supported edges when the temperature varies only along the thickness of the plate. For the case of polygonal shaped plates it is shown that the solution can be deduced from the problem of the torsion of prismatical bars, the cross section of which has the same shape as the plate. The exact solutions for these cases of equilateral triangular and square plates are given. Although only the case of small deflections is considered. its limitations are treated in detail.

LATAYETTE AND GOLDBERG, (Ref. 2) considered the problem of determining the flexural stresses which result from temperature differences between corresponding points on the upper and lower faces of circular plates, when the temperature is assumed to vary only with the distance from the centre of the plate. They have adopted the method of slopes and moments rather than the frontal attack of solving the partial differential equation for the deflection. They have solved the examples of plates with built in, simply supported boundary conditions and plates with elastic edge conditions.

FORRAY AND NEWMAN (Ref. 3) extended the the work of Goldberg by employing the frontal attack method of solving the partial differential equations for the deflection. Solving the problem directly they have presented curves and formulae which are usuful to the practising engineer.

DAS AND NAVARATNA (Ref. 4) have analysed the bending of a rectangular plate with two parallel edges simply supported and the other two edges supported in any manner. The plate is subjected to temperature distribution which is antisymmetric about the middle plane of the plate, but otherwise arbitrary. The boundary conditions are non-homogeneous involving the moment and shear along the edges. The boundary conditions along two parallel edges are made homogeneous and a Levy-type solution is applied for the analysis. Two examples are worked out, to show how to treat various classes of boundary conditions.

NEWMAN AND FORRAY (Ref. 5,6,7) gave a series of notes to present formulae and curves for the deflection, moments and shear in a circular plate with a concentric hole subjected to linear thermal gradient through the thickness. In part I, plate with clamped outer radius coupled with clamped, simple, or free inner boundary are considered. In part II detailed solutions with design curves for a hollow circular plate

simply supported on the outer boundary and simply supported or clamped on the inner boundary is presented. Part III presents the formulae and design curves for the deflections and moments for various cases cited in part I.

NEWMAN AND FORRAY (Ref. 8) presented the nonlinear axisymmetric analysis of plates with in-plane edge restraint. An exact mathematical formulation within the frame work of the Von Karman large-strain displacement relations is developed. The equilibrium equations and boundary conditions are then derived by utilizing the calculus of variations for arbitrary axisymmetric temperatures and normal distributed loading. A finite-difference procedure utilizing "relaxed iterations" has been used for solving the nonlinear ordinary differential equations of this thermomechanical problem. Numarical results are presented for the special case of a simply supported circular plates with radially immovable boundries, subjected to a uniform pressure and an arbitrary temperature variation through the thickness.

SALY GEORGE, HARIHARAN AND GEORGE (Ref. 9) have presented flexural equilibrium equation in cylindrical co-ordinates for uniform circular plates of isotropic,

elastic material under static and thermal loads, using the thick plate shear deformation theory. Neglecting the thermal terms, these equations are solved are solved for deflections and bending moments resulting from axisymmetric static loads. Assuming a temperature distribution which varies linearly through the thickness and arbitrary along the radius of the plate and neglecting the static load, the equations are again solved for thermal deflections and bending moments. The results are presented in the form of nondimensional graphs. The results are also compared with those of the classical theory.

ADIDAM (Ref. 10) presented an exhaustive literature survey and discussed a new method of plastic limit analysis termed "the yield equality method" to analyse a variety of axisymmetric reinforced concrete slabs with different layouts of reinforcement. Design charts were also presented. ADIDAM used this method to design and study the post-yield behaviour of footing slabs subjected to various pressure distributions. He conducted tests on circular slabs with different boundary and loading conditions to prove the validity of the foregoing method.

CHAPTER 3

BENDING MOMENT DUE TO TEMPERATURE GRADIENTS.

3.1 GENERAL

Thermal stresses play an important role in the design of nuclear reactors, gas and steam turbines, aerospace structures and the like. Thermal stresses are generally produced due to

i. non-uniform distribution of temperature

ii. restriction of free strains at the boundaries, and

iii. temperature distribution in a composite structure made of materials having different coefficient of thermal expansions.

Examples for each of these cases are as follows.

A free homogeneous plate having a temperature distribution varing with depth and radius will bend into a curved plate. Thermal stresses will be induced in a circular plate restrained along the circumference and heated uniformly. Thermal stresses are set in a

composite bar of brass and copper heated uniformly.

Strictly speaking, the temperature, stresses and deformations in a solid are interrelated. That is, coupling of temperature and deformation in the exact equations. Thus, the exact analysis must be based on thermodynamic considerations also. This means that the temperature and the stresses must be determined simultaneously. Fortunately, in many practical problems the stresses and deformations have little effect on temperature distribution in the solid. Hence it is customary to assume that the stresses and deformations have no effect on temperature field. Based on this assumption the temperature distribution in the solid is determined independently. In the present work it is assumed that the temperature distribution is known beforehand.

3.2 ASSUMPTIONS

The derivation of basic equations is based on the following assumption.

i. The material is homogeneous isotropic and obeys Hooke's law.

ii. Straight lines initially normal to the middle surface remain straight and normal to

the middle surface after bending and do not change their length.

iii. Effect of normal stress on planes parallel to middle surface of the plate is neglected.

iv. Elastic constants are independent of temperature.

v. Temperature distribution is known before hand.

vi. Temperature changes are gradual and do not depend on time.

vii. Temperature distribution is independent of temperature.

3.3. THERMAL MOMENTS

Thermal moment M_t is given by (Ref. 11)

$$M_t = \frac{E \quad T(z)}{(1 - \nu)} \cdot z \cdot dz \qquad (3.1)$$

3.4.1 TEMPERATURE DISTRIBUTION THROUGH THE THICKNESS.

Let the plate be heated uniformly throughout. Let the temperature at the receiving surface be T_1 and at the other surface T_2. The following temperature distributions are assumed.

a) linear

b) parabolic

and c) exponential

The corresponding thermal moments for these distributions are derived below.

a) Linear distribution (Fig. 1).

Let us consider the curve to be of the form

$$x = az + b$$

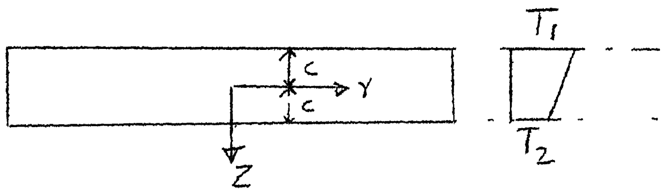


Fig. 1

$$x = T_1 \quad \text{at } z = 0$$

$$x = T_2 \quad \text{at } z = 2c$$

Therefore $$x = \frac{T_2 - T_1}{2c} z + T_1 \qquad (3.2)$$

From equation (3.1) the thermal moment throughout the thickness of the plate will be

$$M_t = \frac{-E\alpha}{(1-\nu)} \int^{+c} \left[\left\{ \frac{T_2 - T_1}{2c} \right\} z + T_1 \right] z dz$$

(3.5)

$$M_t = -\frac{E\alpha c^2}{3(1-\nu)}\left[2T_1 + T_2\right] \qquad (3.4)$$

b) Parabolic distribution (Fig. 2)

Let the curve be of the form

$$x = b - az^2$$

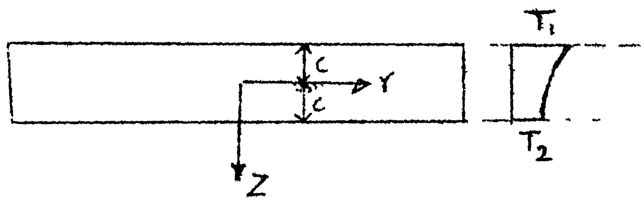


Fig. 2

$$x = T_1 \text{ at } z = 0$$

$$x = T_2 \text{ at } z = 2c$$

$$x = T_1 - \frac{(T_1 - T_2)}{4c^2} z^2 \qquad (3.5)$$

Thermal moment M_t will be

$$M_t = -\frac{E\alpha}{1-\nu}\int_{-c}^{+c}\left[T_1 - \frac{(T_1-T_2)}{4c^2} z^2\right] zdz \qquad (3.6)$$

$$M_t = \frac{E\alpha c^2}{8(1-\nu)}\left[7T_1 + T_2\right] \qquad (3.7)$$

c) Exponential distribution. (Fig. 3)

Let the curve be of the form

as

$$x = b\ e^{a(z-c)} \qquad (3.8)$$

at

$x = T_1 \quad z = -c$

$x = T_2 \quad z = +c$

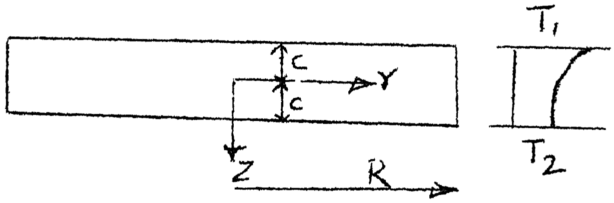


Fig. 3

$$x = T_2\ e^{1/2c \log T_1/T_2 (z-c)} \cdot zdz \qquad (3.9)$$

Thermal moment M_t is

$$M_t = \frac{-E\alpha}{1-\nu} \int_{-c}^{+c} T_2\ e^{1/2c \log T_1/T_2\ (z-c)} \cdot zdz \qquad (3.10)$$

$$M_t = \frac{-2E\alpha}{1-\nu}\ T_2\ \frac{\left(\left(\frac{1}{2} \log \frac{T_1}{T_2} - 1\right) + e^{\frac{1}{2}} \log \frac{T_1}{T_2}\right)}{\left(\frac{1}{2c} \log \frac{T_1}{T_2}\right)^2} \qquad (3.11)$$

3.4.2 RADIAL AS WELL AS THICKNESS VARIATION OF TEMPERATURE.

The distribution of temperature across the thickness of the plate as well as along the radius can be obtained by considering the temperatures at any point the two faces of the plate as the function of the temperature distribution along the radius. Therefore one can express T_1 and T_2 in the equations (3.7), (3.10) and (3.14) in terms of temperature variation along the radius.

Again the three temperature distributions are considered viz.,

a) linear, b) parabolic and c) exponential.

a) Linear Variation (Fig.4)

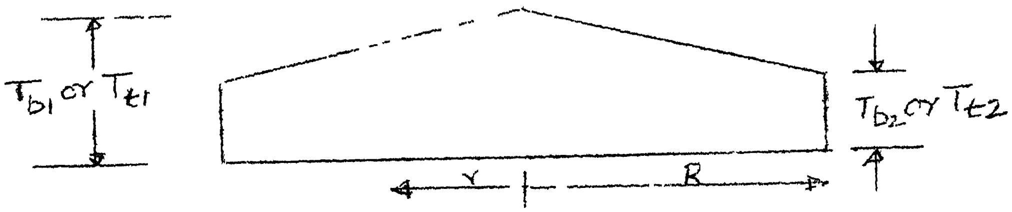


Fig. 4

Radial variation of temperature on the two faces of the plate.

Adopting the same analysis as before one gets for radial variation

$$T_2 = T_{b2} + \left(\frac{R-r}{R}\right) (T_{b1} - T_{b2})$$
$$T_1 = T_{t2} + \left(\frac{R-r}{R}\right) (T_{t1} - T_{t2}) \qquad (3.12)$$

b) Parabolic variation (Fig. 5)

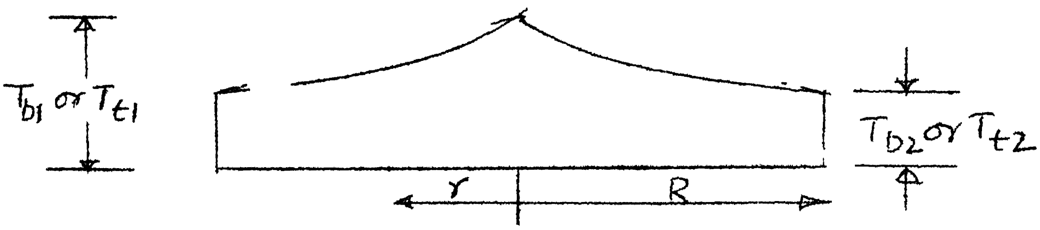


Fig. 5

$$T_1 = T_{t2} + (T_{t1} - T_{t2}) \left(1 - \frac{r^2}{R^2}\right)$$
$$T_2 = T_{b2} + (T_{b1} - T_{b2}) \left(1 - \frac{r^2}{R^2}\right) \qquad (3.13)$$

c) Exponential variation (Fig. 6)

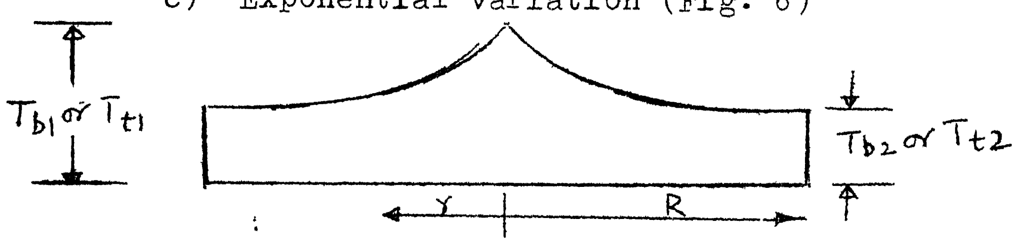


Fig. 6

$$\left.\begin{aligned} T_1 &= (T_{t1} + T_{t2})\, e^{r/R \log\left(\frac{T_{t2}}{T_{t1}+T_{t2}}\right)} \\ T_2 &= (T_{b1} + T_{b2})\, e^{\frac{r}{R} \log\left(\frac{T_{b2}}{T_{b1}+T_{b2}}\right)} \end{aligned}\right\} \qquad (3.14)$$

CHAPTER 4

LIMIT LOAD OF PLATES IN THERMAL ENVIRONMENT

4.1 GENERAL

It is the responsibility of the engineers to design structures to be safe under all working conditions. In this endeavour one should have sufficient margin of safety for unexpected overloads consistent with economy. This has been hither to done by the application of elastic theory. But in recent times application of plastic methods has also been employed to supplement the elastic theory. Plastic theory has the advantage of reflecting the real state of affairs at the collapse stage. This approach is purely behavioural in character and allows the true margin of safety to be assessed. The elastic approach however is necessary to check up deflections under working loads.

4.2 PLASTIC ANALYSIS - BASIC CONCEPTS

Plastic analysis enables one to know the behaviour of the structure loaded upto the plastic range. The generalised stress-strain relationship of any structure is, if at all, linear only in the initial stages of loading;

but at the latter stages, especially at ultimate or collapse stage, it is very complicated. But idealised stress-strain relationships have been assumed in the plastic analysis.

No real material is wholly Hookean (perfectly elastic) or wholly a Saint-Venant substance (rigid perfectly plastic). But they are in general elasto-plastic in character and may be idealised as elastic-perfectly plastic. In respect of steel the stress-strain relationship beyond the workhardening range is neglected and perfect plasticity is assumed. As regards concrete the complicated stress-strain relationship is replaced by an equivalent rectangular stress block. As the deflections during the plastic stage are quite large the initial elastic deflections are ignored in the plastic analysis.

4.2.1 YIELD CONDITION AND FLOW RULE

In the elastic range stress and strain are related by stress-strain relations viz

$$E \epsilon^{(e)} = \sigma , \quad G \gamma^{(e)} = \mathcal{T} \tag{4.1}$$

In the plastic range these relations must be supplemented by yield condition and flow rule. The yield condition expresses all possible combinations of the generalised stresses which produce plastic flow and is also compatible

with the assumption that no plastic flow takes place under a hydrostatic system of stresses. The flow rule gives for any such state of stress the ratios between the increments of plastic strain. The yield condition is expressed as a function that relates the generalised stresses at yield and is given by

$\phi(Q) = 0$ where Q denotes generalised stress

such that

$\phi(Q) < 0$ for no yield.

Yield occurs only when $\phi(Q) = 0$ is satisfied and the combinations of stresses corresponding to $\phi(Q) > 0$ are impossible.

The point, curve or surface corresponding to the above equation is known as the yield point, the yield curve or the yield surface respectively. The yield surface can be shown to be always convex.

The flow rule*is assumed in such a way that the strain vector q is given as

$$q_i = \lambda \frac{d\phi}{dQ_i} \qquad i = 1,2,\ldots n \qquad (4.2)$$

where is an arbitrary factor of proportionality. Above equation implies that the plastic-strain-increment vector is normal to the yield surface at a smooth point and lies

between adjacent normals at a corner.

4.2.2 YIELD CRITERIA

Any number of yield criteria are permissible within the framework of ideal or perfect plasticity. The yield criteria that are described in this section are the Tresca yield criteria, the Von Mises yield criteria and the square yield criteria of Johansen.

TRESCA YIELD CRITERION

It is also known as maximum shearing stress criterion. In terms of principal stresses σ_1, σ_2, for the plane stress case, it is given by

$$\text{Max.}\left(|\sigma_1|, |\sigma_2|, |\sigma_1 - \sigma_2|\right) = \sigma_y \qquad (4.3)$$

This is represented by a Hexagon as shown in the Fig. 4.1 (a)

MISE'S YIELD CRITERION

This was introduced by Mises on mathematical grounds and interpreted later by Hencky & Huber that plastic flow occurs when the shear strain energy stored reaches a definite value. This is represented by an ellipse passing

through the corners of the Hexagon as shown in Fig.4.1(b). Equation of this ellipse is

$$\sigma_1^{3} + \sigma_2^{2} - \sigma_1 \sigma_2 = \sigma_y^{2} \qquad (4.4)$$

SQUARE YIELD CRITERION

This yield criterion was intuitively adopted by Johansen (Ref. 12) in the plastic analysis of reinforced concrete plates. For an isotropic slab in which there is equal amount of reinforcement at top and bottom, in terms of principal stresses the square yield criterion is given by

$$|\sigma_i| = \sigma_y \qquad i = 1,2 \qquad (4.5)$$

The yield cruve and flow rule are shown in Fig. 4.1 (c)

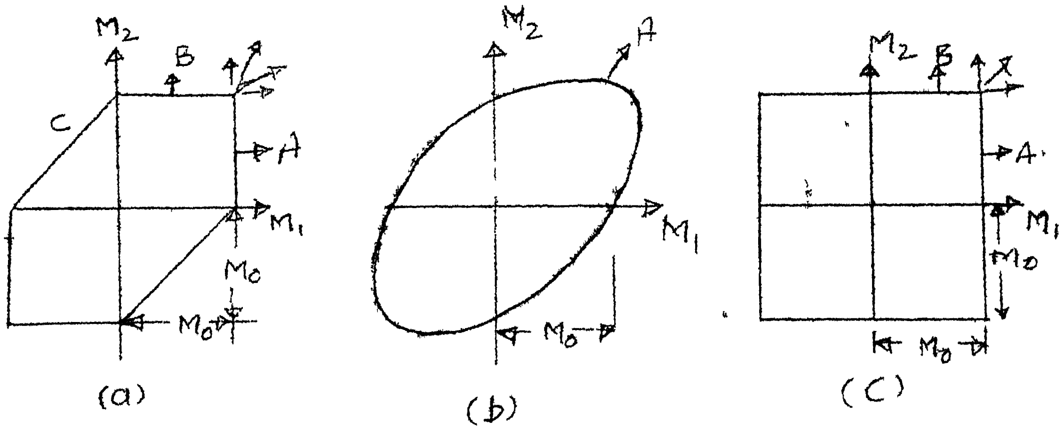


Fig. 4.1 YIELD CRITERIA

4.2.3 FUNDAMENTAL THEOREMS OF LIMIT ANALYSIS

A more general treatment of structures requires a statement of the limit theorems of Drucker, Greenberg and Prager (Ref. 13).

LOWER BOUND THEOREM (PRAGER Ref. 14)

If an equilibrium distribution of stress can be found which balances the applied load and is every where below yield or at yield, the structure will not collapse or will just be at the point of collapse.

UPPER BOUND THEOREM (PRAGER (Ref. 14)

The structure will collapse if there is any compatible pattern of plastic deformation for which the rate at which the external forces do work exceeds the rate of internal dissipation.

The maximum lower bound and minimum upper bound on load capacity are nothing but the limit load itself.

4.3.1 YIELD EQUALITY METHOD

The collapse loads of axisymmetric slabs having axisymmetric loading, reinforcement and boundary conditions are determined in this chapter. The yield line method pioneered by Johansen (Ref.12) has been used extensively for the simpler axisymmetric problems. In the case of non-axisymmetric slabs the correct yield line pattern is known for a few non-trivial cases only. Most of the solutions known are for cases in which total collapse occurs. In such cases, at every point of the slab atleast one yield inequality must be satisfied as a yield equality, since only then does the flow rule of plasticity permit the circumferential curvature rate to be non-zero.

A theorem proposed and proved by Rozvany (Ref.15) provides a unified approach for the determination of collapse loads for all types of collapse mechanisms, whether partial or total. This theorem is based on the lower bound theorem of limit analysis. In an axisymmetric plate with a rectangular yield criterion, let the circumferential plastic moment capacity (yield moment) be $\overline{M}_\theta$ for positive bending and $\alpha \overline{M}_\theta$ for negative bending and the radial yield moments $\overline{M}_r$ & $\beta \overline{M}_r$ respectively. In the theorem that follows, $P(r)$ is the axially symmetric load. $\overline{M}_\theta$, $\overline{M}_r$, α and β are specific

functions of radius r which are piecewise continuous and bounded and the collapse load is $\lambda_0 P(r)$ where λ_0 is the highest statically admissible multiplier.

THEOREM

"The correct load capacity $\lambda_0 P(r)$ of an axisymmetric slab is always associated with at least one piecewise continuous safe statically admissible moment field such that $M_\theta = \bar{M}_\theta$ or $\bar{M}_\theta = -\alpha \bar{M}_\theta$ throughout the slab."

The moment field is said to be statically admissible if it satisfies the equilibrium equation.

$$(r.M_r)'' - M_\theta' = - P(r).r \qquad (4.6)$$

where prime denotes the differentiation with respect to radius r, and it is called safe if it satisfies the yield inequalities.

$$\begin{array}{ccc} -\alpha \bar{M}_\theta & M_\theta & \bar{M}_\theta \\ -\beta \bar{M}_r & M_r & \bar{M}_r \end{array}$$

This theorem by itself does not constitute a sufficient condition for the correct load capacity unless a corresponding kinematically admissible mechanism exists. The existence of such a mechanism can be verified by inspection of the

moment diagram. In the case of partial collapse in rigid regions of the slab an infinite number of safe statically admissible moment fields can be determined with the aid of this theorem. Rozvany gave a proof for the case $\alpha = \beta = 1$, and Rozvany, Charrett, Adidam and Melchers (Ref 15) presented a general proof for any value of α. The general procedure of the yield equality method is outlined in the next section.

4.3.1 PROCEDURE: YIELD EQUALITY METHOD

(a) Select the order (topography) of the regions where $M_\theta = \bar{M}_\theta$ or $M_\theta = -\alpha \bar{M}_\theta$. Region boundaries usually occur along concentrated line loads.

(b) In any region i, calculate the corresponding radial moments using the equilibrium equation (Fig.). In the particular case of M_θ = constant & α = constant, $M'_{\theta i} = 0$, then the radial moment,

$$M_{ri} = \frac{1}{r} \int_A^r \int_A^{\bar{r}} - \bar{\bar{r}} p_i \; (\bar{\bar{r}}) \, . \, d\bar{\bar{r}} \, . \, d\bar{r} \, . \, dr \qquad (4.7)$$

where A is the radius of the inner boundary of the slab, $\bar{r}$ & $\bar{\bar{r}}$ are dummy variables representing radii.

(c) Determine the constants of integration and the load

capacity P from the boundary, continuity and yield conditions.

1. BOUNDARY CONDITIONS

i. In a circular slab at the axis of symmetry, except under a concentrated load,

$$M_r = M_\theta \tag{4.8}$$

ii. At free edges and simple discontinuous supports,

$$M_r = 0 \tag{4.9}$$

iii. At free edges with no concentrated edge loads, the radial shear Q_r is determined by integrating

$$(rM_r)'' - M_\theta' = r \cdot Q_r = 0$$

2. CONDITIONS OF CONTINUITY:

These refer to the continuity of radial moments and shear across a region boundary. The boundary between two adjacent regions i & (i + 1) is termed the region boundary and the radius of the region boundary is denoted by r_i.

At $r = r_i$

i. Moment Continuity:

$$M_{ri} = M_{r(i+1)} \tag{4.10}$$

(4.11)

ii. Shear Continuity:

$$(rM_{ri})' - M_{\theta i} = (rM_{r(i+1)})' - M_{\theta(i+1)} + Q_i \qquad (4.11)$$

where Q_i is the intensity of a concentrated line load, if any, acting along the region boundary.

YIELD CONDITIONS

Some times it is necessary to satisfy the radial yield equalities to obtain a unique moment field, at a finite number of points along a radius of slab.

$$M_r = \bar{M}_r \quad \text{or} \quad M_r = - \beta \bar{M}_r \qquad (4.12)$$

The load capacity can be calculated by solving the the equilibrium equation with the boundary, continuity and yield conditions cited above.

4.4 LIMIT LOAD ANALYSIS OF PLATES WITH THERMAL AND MECHANICAL LOADS.

In this section an attempt is made to include the effect of temperature as thermal load in the equilibrium equation (4.6). It should be noted here that the effect of thermal load on the moment capacity is a function of mechanical load distribution. The thermal moment M_t obtained from the previous chapter is operated upon by ∇^2 to convert it into thermal load term.

Therefore, total thermal load =

$$-\int_0^R 2\pi r \, \nabla^2 M_T \, dr \tag{4.13}$$

The Equilibrium Equation becomes

$$(r\, M_r)'' - (M_\theta)' = - p(r).r - \nabla^2 M_t .r \tag{4.14}$$

CHAPTER V

LIMIT DESIGN OF PLATE ELEMENTS UNDER COMBINED THERMAL AND MECHANICAL COADING.

5.1 GENERAL

From the discussions of the previous chapter one can write the equilibrium equation of the plate subjected to thermal and mechanical loads as

$$(rM_r)'' - M_\theta' = - P.r \qquad (5.1)$$

where P is the combined thermal and mechanical load which is given by,

$$P = p(r) + \nabla^2 M_t$$

For any temperature distribution it can be shown that $\nabla^2 M_t.r$ is a constant (See Appendix).

Let $\nabla^2 M_t.r = K.$

Therefore $$(rM_r)'' - M_\theta' = - p(r).r-K \qquad (5.2)$$

5.2.1 CLAMPED CIRCULAR PLATE WITH UNIFORMLY DISTRIBUTED LOAD.

Consider a circular plate of radius R, clamped along its circumference and subjected to a uniformly distributed load of p and to any constant thermal load K. (Fig. 5.1)

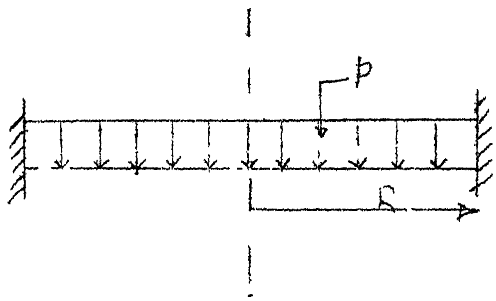


Fig. 5.1

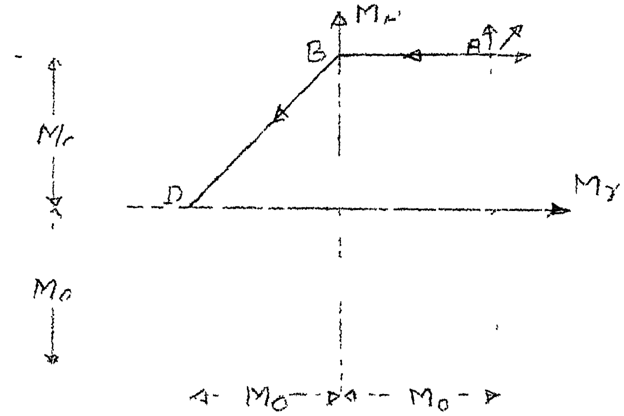
The plate is assumed to obey Tresco's yield criteria. At collapse the plate may be divided into two regions

Region 1 $0 \leq r \leq E$

Region 2 $E \leq r \leq R$

Where E is the radius at which the circumferential moment capacity M_0 changes its sign.

M_θ is constant in region 1 which is equal to M_o. In region 2 it follows the relation $M_\theta = M_r + M_o$ (Fig. 5.2). Therefore the equilibrium



AB - stress path in region 1

BD - stress path in region 2

Fig. 5.2

equations in these two regions will be as follows.

$$(r\, M_{r1})'' = -\ p.r - K$$

$$(r\, M_{r2})'' - M'_{\theta 2} = -\ p.r - K \qquad (5.3)$$

In region 2 since $M_{\theta 2} = M_{r2} + M_o$

$M''_{\theta 2} = M'_{r2}$

Therefore equation 5.3 reduces to

$$(r\, M_{r2})'' - M'_{r2} = -pr - K \qquad (5.4)$$

$(r\, M_{r2})'' - M'_{r2}$ can be written as

$(r\, M'_r)^1$. Therefore, the equilibrium equation in the two regions will be

$$\left.\begin{aligned}(r\, M_{r1})'' &= -\ pr - K\\ (r\, M'_{r2})' &= -\ pr - K\end{aligned}\right\} \qquad (5.5)$$

Therefore

$$\left.\begin{aligned} M_{r1} &= -\frac{pr^2}{6} - \frac{Kr}{2} + C_{11} + \frac{C_{21}}{r} \\ M_{r2} &= -\frac{pr^2}{4} - Kr + C_{12}\log r + C_{22} \end{aligned}\right\} \qquad (5.6)$$

Boundary and continuity conditions are

1. $M_{r1} = + M_0$ at $r = 0$
2. $M_{r2} = 0$ at $r = E$
3. $M_{r1} = M_{r2}$ at $r = E$
4. $M_{r2} = - M_0$ at $r = R$
5. $(rM_{r1})' - M_{\theta 1} = (rM_{r2})' - M_{\theta 2}$ at $r = E$

Applying these conditions to equations (5.6) one gets.

$$M_0 = \frac{PR^2}{12} + \frac{K\,R}{2} \qquad (5.7)$$

This is a closed form solution. So it can be used directly for any problem. When P, R and K are known.

5.2.2 CLAMPED CIRCULAR PLATE WITH TRIANGULAR LOADING.

Consider same plate as in 5.2.1 but with triangular loading as shown in Fig. 5.3

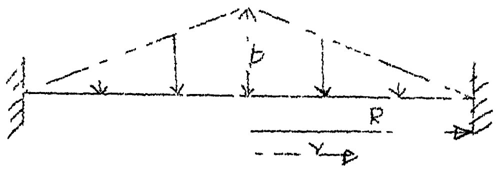


Fig. 5.3

Load distribution in this case will be $p.r = P(1-\frac{r}{R}).r$.
So the final equation of M_r turns out be,

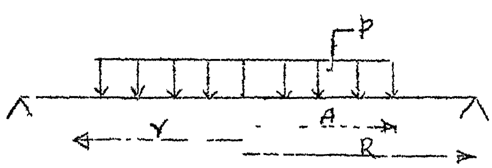
$$\left.\begin{aligned} M_{r1} &= -\frac{Pr^2}{6}\left(1 - \frac{r}{2pR}\right) - \frac{Kr}{2} + C_{11} + \frac{C_{21}}{r} \\ M_{r2} &= -\frac{Pr^2}{6}\left(1 - \frac{r}{2pr}\right) - \frac{Kr}{2} + C_{12} + \frac{C_{22}}{r} \end{aligned}\right\} \qquad (5.8)$$

Applying the same boundary and continuity conditions one gets

$$M_O = \frac{PR^2}{14} + \frac{KR}{4} \qquad (5.9)$$

5.3.1 SIMPLY SUPPORTED PLATE WITH CENTRAL CIRCULAR LOADING.

Consider a plate simply supported along its outer edge, which has a radius R, and with load uniformly distributed over a central circular area of radius A. (Fig. 5.4)



The stress profile must start at point A (A is the point on the stress profile) for $r = 0$ and end at point B for $r = R$ where $M_r = 0$ (Fig. 5.5)

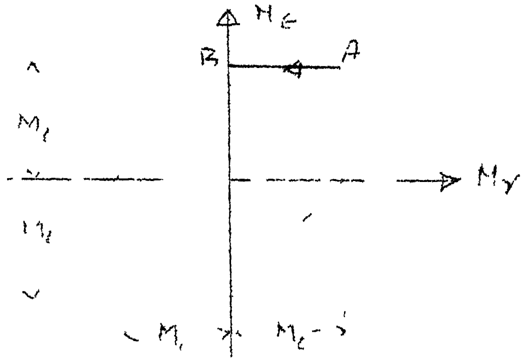


Fig. 5.5

The plate may be divided into two regions

$0 \leq r \leq A$ where $p(r) = p$

$A \leq r \leq R$ where $p(r) = 0$

Therefore the equilibrium equation becomes

$$(r\,M_r)'' - M_\theta' = -\,p.r - K \qquad (5.10)$$

Since M_θ is constant throughout the plate M_θ' is zero.

Therefore $$(r\,M_r)'' = -\,p.r. - K \qquad (5.11)$$

In the region $0 \leq r \leq A$

$$M_{r1} = -\frac{Pr^2}{6} - \frac{Kr}{2} + C_{11} + \frac{C_{21}}{r} \qquad (5.12)$$

In the region $A \leq r \leq R$.

$$M_{r2} = -\frac{Kr}{2} + C_{12} + \frac{C_{22}}{r} \qquad (5.13)$$

Boundary and continuity conditions:

1. $M_{r1} = M_{\theta 1} = M_0$ at $r = 0$
2. $M_{r2} = 0$ at $r = R$
3. $M_{r1} = M_{r2}$ at $r = A$
4. $(r\, M_{r1})' - M_{\theta 1} = (r\, M_{r2})' - M_{\theta 2}$ at $r = A$

Applying these conditions one gets

$$M_0 = \frac{PR^2}{6}\, a^2\,(3-2a) - KR \qquad (5.14)$$

Where $a = \frac{A}{R}$.

$$\frac{M_0}{PR^2} = \frac{a^2\,(3-2a)}{6} - \frac{K}{PR}$$

5.3.2 SIMPLY SUPPORTED PLATE WITH CENTRAL TRIANGULAR LOAD.

Consider the same plate as in 5.3.1 but with central triangular loading. (Fig. 5.6)

Proceeding on the same lines as in the previous case one gets.

$$M_{r1} = -\frac{Pr^2}{6}\left(1 - \frac{r}{3R}\right) - \frac{Kr}{2} + C_{11} + \frac{C_{21}}{r}$$
$$M_{r2} = -\frac{Kr}{2} + C_{12} + \frac{C_{22}}{r} \qquad (5.15)$$

Applying the same boundary and continuity conditions as before one gets

$$M_0 = \frac{PR^2}{6}\, a^2 \left(1 - \frac{a}{2}\right) + \frac{K}{2PR.} \qquad (5.16)$$

$$\frac{M_0}{PR^2} = \frac{a^2}{7}\left(1 - \frac{a}{2}\right) + \frac{K}{2PR}$$

DESIGN EXAMPLES

5.4.1 DESIGN EXAMPLE 1

Consider the temperatures at the top and bottom of the plate as

$T_{t1} = 80^\circ c$ $\qquad$ $T_{t2} = 40^\circ c$

$T_{b1} = 100^\circ c$ $\qquad$ $T_{b2} = 50^\circ c$

Therefore $T_t = 2$ and $T_b = 2$

Let us assume $E = 2 \times 10^6$ Kg / cm^2

$\gamma = 0.000012\ /^{\circ}c.$

$C = 1$ cm.

$R = 30$ cm.

$= 0.5$

Let us take Linear- Linear case of temperature distribution.

$$K = 2\ M_t.r = -16\ Kg/cm.$$

If the plate is clamped and there is u.d.l., Let P = 1 kg. Then, from equation 5.7.

$$M_O = \frac{PR^2}{1^2} + \frac{KR}{2}$$

$$M_O = -165\ kg.\ cm.$$

Therefore the moment capacity of the given plate subjected to given temperature distribution and mechanical loading is - 85 kg. - cm.

5.4.2 DESIGN EXAMPLE 2.

Simply supported plate with triangular loading.

Assuming same type of temperature distribution a as 0.5.

Moment capacity M_O for this case is given by equation (5.

Moment capacity M_0 for this case is given by equation (5.

$$M_0 = \frac{PR^2}{7} a^2 \left(1 - \frac{a}{2}\right) + \frac{KR}{2}$$

$$M_0 = -\ 215.9 \text{ Kg. - cm.}$$

The moment capacity of the plate is -215.9 Kg. cm.

CHAPTER VI

6.1 CONCLUSIONS

The following conclusions can be drawn based on the foregoing investigation.

1. For a plate clamped or simply supported along its boundary and subjected to any type of temperature distribution and uniformly distributed load the percentage reduction in the moment capacity due to thermal load is 16.67.

2. For a plate clamped or simply supported along its boundary and subjected to any type of temperature distribution and subjected to triangular loadings the percentage reduction in the moment capacity is 28.57.

3. It is seen that for a given loading condition the percentage reduction of moment capacity due to thermal load is same irrespective of the support conditions. This justifies the statement made in chapter , that thermal load distribution is a function of mechanical load distribution.

4. The solutions obtained for the moment capacity are closed form solutions and such they can be used directly to find the moment capacity of a given plate.

6.2 SCOPE FOR FUTURE WORK

The present work can be extended to annular plates, plates with thermal inclusions and tapered plates. Optimisation techniques can be used to arrive at optimum taper angles for tapered plates.

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APPENDIX

TO SHOW THAT THE THERMAL LOAD DISTRIBUTION IS CONSTANT.

Let us consider any combination of temperature distribution. Say linear-linear.

From chapter (3) one can write, for the above temperature distribution,

$$M_t = \frac{E \alpha c^2}{3(1-\nu)} \left[2T_{t2} \left[1 + \left(\frac{R-r}{R}\right)\left(\frac{T_{t1}}{T_{t2}} - 1\right) \right] - T_{b2} \left[1 + \left(\frac{R-r}{R}\right)\left(\frac{T_{b1}}{T_{b2}} - 1\right) \right] \right].$$

$$\nabla^2 M_t = \frac{E \alpha c^2}{3R\, r(1-\nu)} \left[T_{b2} (T_b - 1) - 2T_{t2}(T_t - 1) \right]$$

where $T_b = \dfrac{T_{b1}}{T_{b2}}$ and $T_t = \dfrac{T_{t1}}{T_{t2}}$

The only variable here is r. While taking thermal load distribution,

One gets $\nabla^2 M_t . r$ = Constant.

Similarly it can shown that $\nabla^2 M_t . r$ remains constant for any type of temperature distribution.